Zuse's Z3 Square Root Algorithm

Talk given at Fall meeting of the Ohio Section of the MAA
October 1999 - College of Wooster
(Revised 02/02/2009)

Abstract

Brian J. Shelburne
Dept of Math and Comp Sci
Wittenberg University

In 1941 Konrad Zuse completed the Z3 sequence controlled calculator. The Z3 was a binary machine (numbers were stored internally in floating point binary) with a 64 by 22 bit memory which could perform the four standard arithmetic operations as well as take square roots. This talk examinations the algorithm Zuse used for his square root instruction and how it was implemented on the Z3.

Zuse's Z3 Square Root Algorithm- The Talk

Konrad Zuse (1910 - 1995)

born 1910 in Berlin, Germany
studied civil engineering at Technische Hochschule Berlin-Charlottenburg
worked as design engineer in aircraft industry
needed calculations for design and analysis of structures
his first attempt at automating calculation process was to design a form to track intermediate results of calculations
not familiar with design of mechanical calculators

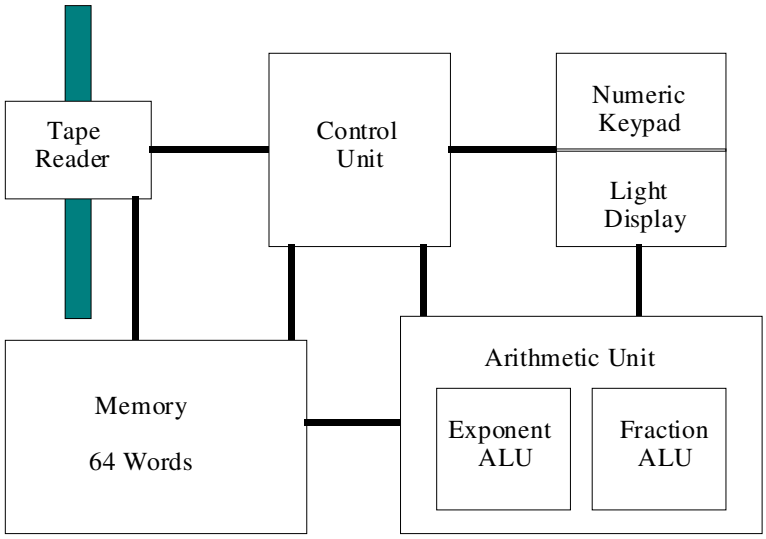
His Early Machines: The Z1 - Z3

Z1: begun in 1936
floating-point binary representation
24 bit word length - 1 sign bit, 7 bit exponent, 16 bit mantissa (fraction)
4 decimal digit precision
completely mechanical (memory, arithmetic unit and control)
16 word prototype memory completed 1937
controlled by tape (used discarded 35 mm film) - hand punched!
four functions : + - × ÷
mechanical arithmetic unit unreliable

Z2: 1938 - 1940 (smaller hybrid machine)
relay based ALU and control; mechanical memory
"proof of concept" for relay based calculator
demonstration led to funding for Z3 by Deutsche Versuchsanstalt fur Luftfahrt

Z3: 1939 - 1941 (relay based machine)
same design as Z1 except *included* square root operation

An Overview of the Z3



Memory : 64 by 22 bits - normalized floating point binary

```
|_|_ _ _ _ _ _ _|_ _ _ _ _ _ _ _ _ _ _ _ _ _ _|
 ±     exponent              fraction
       a6 … a0              b-1  ...  b-14
```

Numeric Keyboard input: 4 columns of 10 keys for digits
1 row of exponent keys labeled: -8, -7 ... 7, 8

Display Output - row of 5 lights for fraction plus exponent lights labeled: -8, -7 ...7, 8

Tape Control: nine 8 bit instructions

```
01 110000 - read keyboard          01 100000 - add
01 111000 - display result         01 101000 - subtract
                                   01 001000 - multiply
11 z1z2z3z4z5z6 - load address     01 010000 - divide
10 z1z2z3z4z5z6 - store address    01 011000 - square root
```

Arithmetic Unit: separate ALU's for exponents and fractions

Control Unit: instructions were implemented as a sequence of micro-operations which controlled movement of data through machine.

How Do You Take A Square Root?

1. Use a calculator

2. Use a slide rule

3. Use a table of logarithms

4 Use Newton's Iteration Formula to solve $f(x) = x^2 - a$ by using the iteration formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{1}{2} \times (x_n - \frac{a}{x_n})$ where x_0 equals some initial "guess" ($x_0 = a/2$ works well for $a > 1$). Note this requires one division per iteration (expensive!).

5. Use the binomial expansion of $\sqrt{1-x}$

$$\sqrt{1-x} =$$
$$1 - \frac{1}{2}x^1 - \sum_{k=2}^{\infty} \frac{1 \times 3 \times 5 \times ... (2k-3)}{2^k \times k!} x^k =$$
$$1 - \frac{1}{2}x^1 - \frac{1}{2^2 \times 2!}x^2 - \frac{1 \times 3}{2^3 \times 3!}x^3 - \frac{1 \times 3 \times 5}{2^4 \times 4!}x^4 - ..$$

We can calculate the $\sqrt{a}$ by finding some value b^2 close to a and using the above series expansion for $x = 1 - \frac{a}{b^2}$ since

$$\sqrt{a} = \sqrt{b^2 + a - b^2} = b\sqrt{1 - (1 - \frac{a}{b^2})}$$

6. Complete the Square: This technique is based on the simple idea that if x is a "good" approximation to $\sqrt{a}$ where $a - x^2 = e > 0$ is the error, then find a small value b such that $(x+b)$ is a better approximation; that is $a - x^2 > a - (x+b)^2 = (a - x^2) - 2bx - b^2 \geq 0$ or to put another way, find a value b such that $e - b(2x + b) \geq 0$. The inequality is not linear in b but is it not difficult to *questimate* a value. Generally this is done digit by digit.

Completing the Square (details)

Without loss of generality assume $1 \leq a < 100$ and suppose $x = Q_{k-1}$ is an approximation to $\sqrt{a}$ accurate to $10^{-(k-1)}$ (i.e. the $k-1^{st}$ digit *below* the decimal point). That is

$$Q_{k-1} = d_0 + \sum_{i=1}^{k-1} d_i \times 10^{-i} \text{ where } d_i \text{ is a digit between 0 and 9}$$

Letting $b = d_k 10^{-k}$, find the largest digit d_k such that

$$a-(x+b)^2 = a-\left(Q_{k-1}+d_k 10^{-k}\right)^2 = \left(a-{Q_{k-1}}^2\right)-2Q_{k-1}\left(d_k 10^{-k}\right)-\left(d_k 10^{-k}\right)^2 \geq 0$$

To find the largest such digit d_k factor out $d_k 10^{-k}$ to obtain

$$\left(a-Q_{k-1}\right)^2 - d_k 10^{-k}\left(2Q_k + d_k 10^{-k}\right) \geq 0$$

Scaling (multiply) by 10^{2k} does not change the sign of the expression but it makes all values integers.

$$10^{2k}\left(a-{Q_{k-1}}^2\right) - d_k \times \left(2Q_{k-1}\times 10^k + d_k\right) \geq 0$$

To find the largest digit d_k use the following *guess and check* method.

1. Guess the *largest digit* d_k such that $d_k \times 2Q_{k-1}\times 10^k$ is less than $10^{2k} e = 10^{2k}\left(a-{Q_{k-1}}^2\right)$
2. Check that $d_k \times \left(2Q_{k-1}\times 10^k + d_k\right)$ is still less than the "scaled error". If not reduce by 1 and try again.

If successful then $Q_k = d_0 + \sum_{i=1}^{k} d_i 10^{-i}$ is a better approximation with a smaller "scaled error" now equal to

$$10^{2k}\left(a-Q_{k-1}\right)^2 - d_k \times \left(2Q_{k-1}\times 10^k + d_k\right) = 10^{2k}\left(a-Q_k\right)^2 < 10^{2k}\left(a-Q_{k-1}\right)^2$$

Finding the largest d_k which maintains this inequality is the theory behind the following well-known algorithm (well-known before the advent of cheap calculators) used to compute a square root by hand. To demonstrate we take the square root of 20.375.

```
                    d0  d1 d2  d3
                    4.  5  1  3
                  √20. 37 50 00 <- a
                   16  00       <- 10^2 × Q0^2
                   ------
4×20+d1 ->   85      4  37       <- 10^2 × (a - Q0^2)
                    4  25       <-  d1 × (2Q0 × 10 + d1)
                    ------
45×20+d2 -> 901        12 50      <- 10^4 × (a - Q1^2)
                        9 01      <-  d2 × (2Q1 × 10^2 + d2)
                       -----
                        3 49 00      <- 10^6 × (a - Q2^2)
451×20+d3 -> 9023       2 70 69      <-  d3 × (2Q2 × 10^3 + d3)
                       --------
                          78 31 00
```

First - find the largest integer $d_0 = Q_0$ whose square is less than or equal to the integer part of the number *a*. Square Q_0 and subtract it from the integer part. Bring down the next two digits.

Second. - Find the largest integer d_1 such that $2 \times Q_0 \times 10 + d_1$ times d_1 is less than the previous remainder. Subtract it and bring down the next two digits. $Q_1 = Q_0 + d_1 10^{-1}$

Third - Find the largest integer d_2 such that $2 \times Q_1 \times 10^2 + d_2$ times d_2 is less than the previous remainder. Subtract it and bring down the next two integers. $Q_2 = Q_1 + d_2 10^{-2}$

k[th] Step - Find the largest integer d_k such that $2 \times Q_{k-1} \times 10^k + d_k$ times d_k is less than the previous remainder. Subtract it and bring down the next two integers. $Q_k = Q_{k-1} + d_k 10^{-(k-1)}$ Repeat.

The Binary Version of the "Completing the Square" Square Root Algorithm

The same algorithm can be easily done in binary. By way of demonstration, we find the square root of the binary value 10100.011 which is 20.375 decimal. The floating point representation is 1.0100011×2^{100} (note the even exponent). Therefore the square root of 10100.011 is the square root of 1.0100011 times 2^{10} (or 4).

Remember that multiplication by 2 in binary is equivalent to a left shift. Furthermore scaling by powers of 10 (binary) is scaling by powers of 2.

```
       1 .  0  0  1  0  0  0  0  0  1  1
      √1 . 01 00 01 10 00 00 00 00 00 00
       1
       -
       0   01         <- 2^2 × (a - Q0^2)
100    0   00         <- b1 × (2 × Q0 × 2^1 + b1)
       ------
            1 00<- 2^4 × (a - Q1^2)
1000        0 00<- b2 × (2 × Q1 × 2^2 + b2)
            ----
            1 00 01  <- etc...
10001       1 00 01
            -------
                  0 10
100100            0 00
                  ----
                    10 00
1001000             00 00
                    -----
                    10 00 00
10010000            00 00 00
                    ----------
                    10 00 00 00
100100000           00 00 00 00
                    -----------
                    10 00 00 00 00
1001000000          00 00 00 00 00
                    --------------
                    10 00 00 00 00 00
10010000001          1 00 10 00 00 01
                    ----------------
                       11 01 11 11 11 00
100100000011           10 01 00 00 00 11   whew !!!
```

So therefore the square root of 10100.011 is approximately $1.0010000011 \times 2^{10} = 100.10000011 =$ 4.51171875 which is "close" to 4.513867521

The Z3 Square Root Implementation - Completing the Square in Binary

The Z3 used a variant of the above algorithm to compute a square root. Given $1 \le a < 100_2$ (values are in binary) if Q_{k-1} is an approximation to $\sqrt{a}$ accurate to $2^{-(k-1)}$, that is

$$Q_{k-1} = b_0 + \sum_{i=1}^{k-1} b_i 2^{-i}$$

where each b_i is either 0 or 1, then find the *largest* binary digit b_k (0 or 1) such that

$$a - \left(Q_{k-1} + b_k 2^{-k}\right)^2 = \left(a - {Q_{k-1}}^2\right) - 2 \times Q_{k-1} \times \left(b_k 2^{-k}\right) - \left(b_k 2^{-k}\right)^2 \ge 0$$

Because there are only two possible choices for b_k , either 0 or 1, this algorithm is easy to implement. Moreover since ${b_k}^2 = b_k$ these are some "tricks" to make iteration efficient. Begin by scaling (multiplying) the above inequality by 2^k (instead of 2^{2k}). If we have

$$2^k \times \left(a - {Q_{k-1}}^2\right) - \left(2 \times Q_{k-1} + 2^{-k}\right) \ge 0$$

then b_k is 1; otherwise $b_k = 0$. In any case since ${b_k}^2 = b_k$ it follows that the above expression $2^k \times \left(a - {Q_{k-1}}^2\right) - \left(2 \times Q_{k-1} + 2^{-k}\right)$ equals $2^k \left(a - \left(Q_{k-1} + b_k 2^{-k}\right)^2\right) = 2^k \left(a - {Q_k}^2\right)$ which can be used in the calculation for the next step. If we multiply $2^k \left(a - {Q_k}^2\right)$ by 2 we obtain $2^{k+1} \left(a - {Q_k}^2\right)$ for use in the next iteration step. Multiplication by 2 is easily done on a binary machine using a left shift.

The algorithm is given below. Assume $1 < a \le 100_2$ and let $e_k = 2^k \left(a - {Q_k}^2\right)$ (a *scaled error value*) where $e_0 = (a - 1)$.

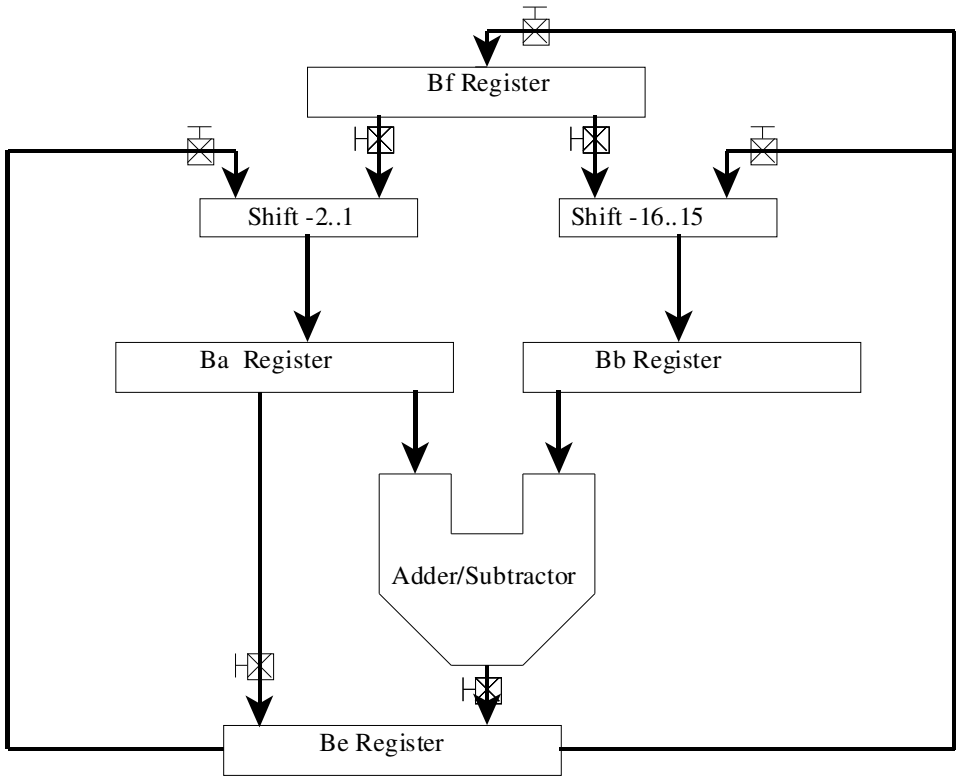
1. let Q[0] = 1
2. let e[0] = a - 1
3. for k = 1, 2, 3, . . .
   - 3.1 Compute $2 \times e[k-1] - (2 \times Q[k-1] + 2^{-k})$
   - 3.2 If positive or zero (i.e. b[k] = 1)
     - 3.2.1 $Q[k] = Q[k-1] + 2^{-k}$
     - 3.2.1 $e[k] = 2 \times e[k-1] - (2 \times Q[k-1] + 2^{-k})$
   - 3.3 Otherwise (i.e. b[k] = 0)
     - 3.3.1 Q[k] = Q[k-1]
     - 3.3.2 $e[k] = 2 \times e[k-1]$

Example - Compute the square root of 1.0100011

k	**$2\times e[k-1] - (2 \times Q[k-1] + 2^{-k})$**		**$e[k]= a - Q[k]^2$**	**Q[k]**
0			0.0100011	1.
1	0.100011 - 10.1	negative	0.100011	1.0
2	1.00011 - 10.01	negative	1.00011	1.00
3	10.0011 - 10.001	positive	0.0001	1.001
4.	0.001 - 10.0101	negative	0.001	1.0010
5.	0.01 - 10.01001	negative	0.01	1.00100
6	0.1 - 10.010000	negative	0.1	1.001000
7	1.0 - 10.0100001	negative	1.0	1.0010000
8	10.0 - 10.01000001	negative	10.0	1.00100000
9	100.0 - 10.010000001	positive	1.101111111	1.001000001
10	11.01111111 - 10.010000011	positive	1.001111011	1.0010000011

This agrees with the example given above.

A Closer Look at the Architecture of the Z3 Arithmetic Unit



Z3 had separate ALU's to handle the exponent and fractional parts of a number

Basic data path of the fraction ALU consisted of four registers (Bf, Ba, Bb, and Be) an addition/subtraction unit and two shifter units connected by data lines. A number of gates controlled the follow of data around the data path

Hardware operations included **addition** and **subtraction**, pre-**shifting** input to the Ba and Bb registers, **detecting a positive or zero result** and **setting individual bits** in a register.

A *micro-operation* consisted of a timed sequence of gates openings that allowed the contents of registers to flow through the pre-shifters and addition/subtraction unit. Detecting a positive or zero result could conditionally alter the flow of data.

Instructions were implemented as *sequences* of micro-operations.

Micro-operation Implementation of Z3 Square Root

Note Bf[-k] refers to bit -k in register Bf

0.	Bf := a	// Load Bf with a
	Ba := Bf (or $2 \times$ Bf if exponent is odd)	// Ba = a
	Bb[0] := 1	// $Q_0 = 1$
1.	if Ba - Bb $\geq$ 0 then	// test
	Be := Ba - Bb	
	Bf[0] := 1	// set bit
	else	
	Be := Ba	
		// set up next comparison
	Ba = $2 \times$ Be	// $2 \times r_0$
	Bb = $2 \times$ Bf; Bb[-1] := 1	// $2 \times Q_0 + 2^{-1}$
2	if Ba - Bb $\geq$ 0 then	// test
	Be := Ba - Bb	
	Bf[-1] := 1	// set bit
	else	
	Be := Ba	
		// set up next comparison
	Ba := $2 \times$ Be	// $2 \times r_1$
	Bb := $2 \times$ Bf; Bb[-2] := 1	// $2 \times Q_1 + 2^{-2}$

and in general

	if Ba - Bb $\geq$ 0 then	// if $2r_{k-1} - (2 \times Q_{k-1} + 2^{-k}) \geq 0$ then
	Be := Ba - Bb	// $r_k = 2 \times r_{k-1} - (2 \times Q_{k-1} + 2^{-k})$
	Bf[-k] := 1	// $Q_k = Q_{k-1} + 2^{-k}$
	else	
	Be := Ba	// $r_k = 2 \times r_{k-1}$
		// $Q_k = Q_{k-1}$
		// set up next comparison
	Ba := $2 \times$ Be	// $2 \times r_k$
	Bb := $2 \times$ Bf; Bb[-(k+1)] := 1	// $2 \times Q_k + 2^{-(k+1)}$

At the end the exponent which was even was divided by 2 (a right shift) yielding the square root.

The thing to notice was the *economy of operations* needed to compute a square root - shifts, subtractions, comparisons and setting individual bits within a word.

Summary

1. The square root algorithm used by the Z3 was the binary analog of the "completing the square" algorithm, a widely known (?) manual technique for obtaining a square root one digit at a time.

2. For $1 \le a < 100_2$ and Q_{k-1} an approximation of $\sqrt{a}$ accurate to $k^{-(k-1)}$, set bit b_k equal to 1 if

$$a - \left(Q_{k-1} + b_k 2^{-k}\right)^2 \ge 0$$

otherwise set bit b_k equal to 0. It was shown that this is the same as checking if

$$2^k \left(a - Q_{k-1}{}^2\right) - \left(2Q_{k-1} + 2^{-k}\right) \ge 0$$

and leads to the iteration algorithm

Repeat until tired
Compute $2 \times e[k-1] - (2 \times Q[k-1] + 2^{-k})$
If positive or zero
    $Q[k] = Q[k-1] + 2^{-k}$
    $e[k] = 2 \times e[k-1] - (2 \times Q[k-1] + 2^{-k})$
Otherwise
    $Q[k] = Q[k-1]$
    $e[k] = 2 \times e[k-1]$

where $e[k] = (a - Q[k]^2)$ and initial condition $Q_0 = 1$ and $e[0] = (a - 1)$

3. A close examination of this algorithm reveals that it requires only the operations of **subtraction**, **left shift** (multiplication by 2), **set bit**, and **test if positive or zero**. - all of which are easy to implement in binary logic.

4. The Z3 had a data path that could easily implement these operations and a micro-operation sequencer that could iterate each step of the algorithm the requisite number of times. Use of floating point binary notation helped.

5. Binary representation is used in computers because for economic reasons computer hardware is built out of bi-stable components. However, many complicated arithmetic operations like multiplication, division and square root are easy to implement in binary. Even though we humans find binary notation complicated to use (even moderately sized numbers take a lot of bits), because of the ease in implementing complicated algorithms in binary, there is something to be said for the advantages of binary notation in and of itself.

Any sufficiently advanced civilization uses a number system based on a power of two.

Epilog: A Follow Up on the Z3 - the Z4

The Z3 was really a "proof of concept" for the more ambitious Z4, a 32 bit machine begun after the Z3 was completed. Unlike the Z3, the Z4 used a mechanical memory of 1000 words (which took up a cubic meter of space!). Before it was completed it was moved to Gottingen to avoid the Berlin air raids which had destroyed the Z3.

Just before Gottingen was "liberated" at the end of WWII, the Z4 was moved to Hinterstein, Bavaria. In 1950 the refurbished Z4 with some additional features that gave it a conditional jump was moved to ETH Zurich. In 1955 it was moved to Basel where it gave "acceptable service" until 1960.

After the war Zuse founded Zuse Kommandit Gesellschaft (Zuse KG) which produced for the Leitz Optical Company a relay based, punch-tape controlled "computer" called the Z5 which was based on the Z4 design. This led to the production of a number of relay-based calculators, called the Z11. In 1956 Zuse began work on a vacuum-tube based computer called the Z22 with a transistor version following. A series of mergers and takeovers finally lead to Zuse KG becoming part of Siemens.

References

R. Rojas, "Konrad Zuses' Legacy: The Architecture of the Z1 and Z3", Annals of the History of Computing, vol. 19, no. 2, pp. 5 - 16, 1997

I. Koran, *Computer Arithmetic Algorithms*, Englewood Cliff, N.J.: Prentice-Hall, 1993

M. Williams, *A History of Computing Technology*, Los Alamitos, CA: IEEE Computer Society Press, 1997